Proofs Without Syntax
Part I: Some Examples

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We exhibit combinatorial proofs of some propositions. Recall that for every proposition ϕ we can build recursively its associated graph, denoted by $G(\phi)$. The kernel of the map $G(-)$ includes the relation $\cong$, which denotes congruence modulo associativity and commutativity of $\wedge$ and $\vee$, double negation ($\neg\neg\phi \cong \phi$), de Morgan duality and which is closed under $\phi \to \psi \cong \neg\phi \vee \psi$. We simplify every proposition until we arrive to a form which is equivalent under $\cong$ and easier to depict.
To give a combinatorial proof of ϕ is tantamount to give a map

$$h : C \to G(\phi)$$

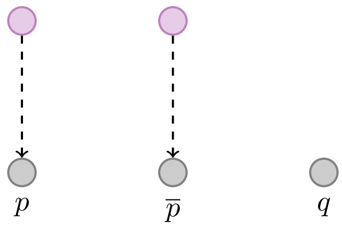
where

- C is nicely coloured, *which means*
  - every colour class of C has at most two vertices
  - no union of two-vertex colour classes induces a matching
- C is a cograph
- every colour class of C is axiomatic
- h is a skew fibration (in particular, h is a a graph homomorphism)

We will not check that those conditions hold but the reader may easily verify that it does.

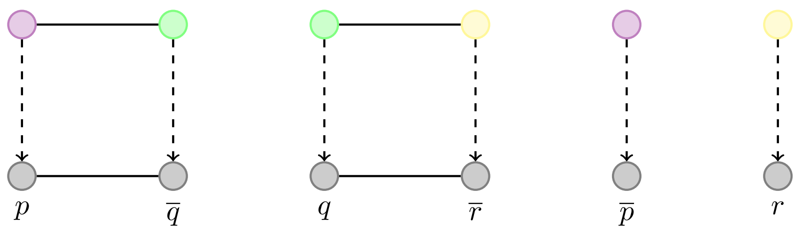
a) $\neg p \to (p \to q)$. Observe that

$$\begin{aligned} \neg p \to (p \to q) &\cong \neg\neg p \vee (p \to q) \\ &\cong p \vee \neg p \vee q \end{aligned}$$



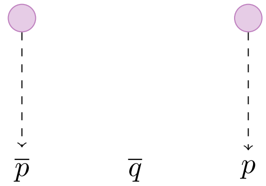
b) $(p \to q) \to [(q \to r) \to (p \to r)]$. Observe that

$$\begin{aligned}(p \to q) \to [(q \to r) \to (p \to r)] &\cong \neg(p \to q) \vee [(q \to r) \to (p \to r)] \\ &\cong (p \wedge \neg q) \vee \neg(q \to r) \vee (p \to r) \\ &\cong (p \wedge \neg q) \vee (q \wedge \neg r) \vee \neg p \vee r\end{aligned}$$



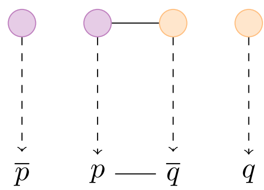
c) $p \to (q \to p)$. Observe that

$$\begin{aligned}p \to (q \to p) &\cong \neg p \vee (q \to p) \\ &\cong \neg p \vee \neg q \vee p\end{aligned}$$



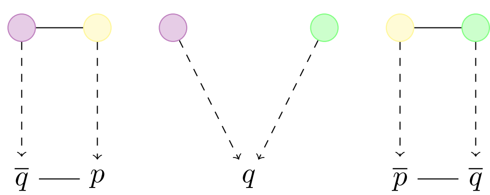
d) $p \to [(p \to q) \to q]$. Observe that

$$\begin{aligned} \neg p \vee [(p \to q) \to q] &\cong \neg p \vee \neg(p \to q) \vee q \\ &\cong \neg p \vee (p \wedge \neg q) \vee q \end{aligned}$$



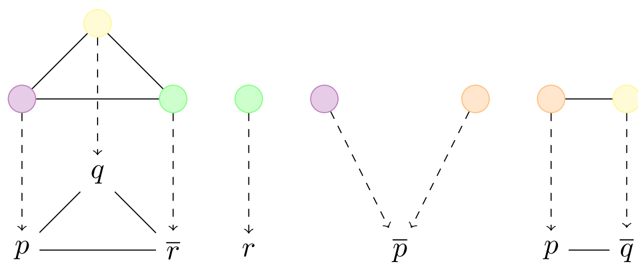
e) $[(p \to q) \wedge (\neg p \to q)] \to q$. Observe that

$$\begin{aligned} [(p \to q) \wedge (\neg p \to q)] \to q &\cong \neg[(p \to q) \wedge (\neg p \to q)] \vee q \\ &\cong \neg(p \to q) \vee \neg(\neg p \to q)] \vee q \\ &\cong (p \wedge \neg q) \vee (\neg p \wedge \neg q) \vee q \end{aligned}$$



f) $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$. Observe that

$$\begin{aligned} [p \to (q \to r)] \to [(p \to q) \to (p \to r)] &\cong \neg[p \to (q \to r)] \vee [(p \to q) \to (p \to r)] \\ &\cong [p \wedge \neg(q \to r)] \vee \neg(p \to q) \vee (p \to r) \\ &\cong (p \wedge q \wedge \neg r) \vee (p \wedge \neg q) \vee \neg p \vee r \end{aligned}$$


q
p
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